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Generalized two-mode harmonic oscillator: $SO(3,2)$ dynamical group and squeezed states

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Abstract. In this paper we generalize our previous results of the generalized one-mode harmonic oscillator to the generalized two-mode case. Systematic use is made of the $SO(3,2)$ dynamical group and we are able to write a general form for the exact time evolution operator in terms of squeezing operators of one and two modes. A complete classification of the exact solutions is made and we derive them explicitly whenever possible. The relevant results on algebraic decomposition, coherent state generators and classification of the solutions are shown in tables. A plethora of soluble Hamiltonians already treated in the literature, which appear to be particular cases of the general formalism presented herein, are analysed as well as new cases, which to the authors' knowledge, have not yet been considered.

1. Introduction

Two-mode time-dependent oscillators are the origin of an important body of modern scientific literature concerning applications to quantum optics, squeezing and laser interactions in two-level atoms. We shall be dealing with this problem in a systematic way in this paper such that a complete classification of the quantum integrable cases shall be given, taking advantage of the relationship with the dynamical group $SO(3,2)$.

Let us consider here the generalization to two degrees of freedom of the one-mode time-dependent harmonic oscillator which was considered extensively by the present authors in [1]. The generalized two-mode Hamiltonian is now given by

$$H(t) = z_1\frac{p_1^2}{2m} + \frac{\omega_1}{2}u_1(x_1p_1 + p_1x_1) + \frac{\omega_1^2}{2}mv_1x_1^2 + z_2\frac{p_2^2}{2m} + \frac{\omega_2}{2}u_2(x_2p_2 + p_2x_2) + \frac{\omega_2^2}{2}mv_2x_2^2 + z\frac{p_1p_2}{2m} + \sqrt{\omega_1\omega_2}(ux_1p_2 + u'x_2p_1) + \frac{1}{2}mv\omega_1\omega_2x_1x_2 \qquad (1.1)$$

where $z(t), z_1(t), z_2(t), u(t), u'(t), u_1(t), u_2(t), v(t), v_1(t)$ and $v_2(t)$ are real functions of time such that $H(t)$ remains Hermitian and verify the initial conditions which allow us to reduce the system to two initially uncoupled one-dimensional harmonic oscillators. In the literature we have found several particular cases ($u = u' = z_1 = z_2 = v_1 = v_2 = 0$) [2] or ($z_1 = z_2 = 1$ and $u = u' = u_1 = u_2 = z = 0$) [3, 4]. It has also been used in [5] to describe the interaction of a charged two-dimensional oscillator under the action of a magnetic field. In [6] linear terms were added to this Hamiltonian which can also be treated easily by means of our formalism.

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The introduction of the bosonic operators a_1, a_2

$$a_1 = \frac{1}{\sqrt{2\hbar m\omega_1}}(m\omega_1 x_1 + \mathrm{i}p_1) \tag{1.2}$$

$$a_2 = \frac{1}{\sqrt{2\hbar m\omega_2}}(m\omega_2 x_2 + \mathrm{i}p_2) \tag{1.3}$$

that satisfy the canonical conmutation relation

$$[a_i, a_j] = [a_i^+, a_j^+] = 0 \qquad [a_i, a_j^+] = \delta_{ij} \qquad i, j = 1, 2 \tag{1.4}$$

and allow us to write the Hamiltonian as

$$H(t) = 2s_1\left(a_1^+a_1 + \tfrac{1}{2}\right) + g_1^* a_1^2 + g_1 a_1^{+2} + 2s_2\left(a_2^+a_2 + \tfrac{1}{2}\right) + g_2^* a_2^2 + g_2 a_2^{+2} + 2d a_1^+ a_2^+ + 2d^* a_1 a_2 + 2e a_1^+ a_2 + 2e^* a_1 a_2^+ \tag{1.5}$$

with

$$g_j = \frac{\hbar\omega_j}{4}[(v_j - z_j) + 2\mathrm{i}u_j] \qquad j = 1, 2 \tag{1.6}$$

$$s_j = \frac{\hbar\omega_j}{4}(v_j + z_j) \qquad j = 1, 2 \tag{1.7}$$

$$2d = \frac{\hbar\sqrt{\omega_1\omega_2}}{4}(v - z) + \mathrm{i}\frac{\hbar}{2}(\omega_2 u + \omega_1 u') \tag{1.8}$$

$$2e = \frac{\hbar\sqrt{\omega_1\omega_2}}{4}(v + z) - \mathrm{i}\frac{\hbar}{2}(\omega_2 u - \omega_1 u')\,. \tag{1.9}$$

This new representation, in which the main role is now played by the canonical operators (a_1, a_1^+, a_2, a_2^+), clarifies the interpretation of the system as describing the normal modes of the quantized electromagnetic field. This is the main reason why $H(t)$ can be (and actually is being) used in quantum optics where the description of nonlinear quantum interactions between light and different kinds of optical material media makes the formalism very useful by looking at the interaction of photons as the active part of the dynamics while the optical media is regarded as the passive part of the system.

2. The Lie algebra of the group $SO(3, 2)$

The identification of the dynamical symmetry of the present system [1–3] requires us to consider the bilinear products of two independent bosonic operators. Given two bosonic operators a_1, a_2 that satisfy (1.4), we can construct the Hermitian operators

$$J_{12} = \tfrac{1}{2}[a_1^+a_1 - a_2^+a_2] \qquad J_{15} = \tfrac{1}{4}[a_1^{+2} + a_1^2 - a_2^{+2} + a_2^2] \tag{2.1}$$

$$J_{13} = \tfrac{1}{2}[a_1^+a_2 - a_1a_2^+] \qquad J_{23} = \tfrac{1}{2}\mathrm{i}[a_1^+a_2 - a_1a_2^+] \tag{2.2}$$

$$J_{14} = \tfrac{1}{4}\mathrm{i}[a_1^{+2} - a_1^2 + a_2^{+2} - a_2^2] \qquad J_{24} = -\tfrac{1}{4}[a_1^{+2} + a_1^2 - a_2^{+2} - a_2^2] \tag{2.3}$$

$$J_{25} = \tfrac{1}{4}\mathrm{i}[a_1^{+2} - a_1^2 - a_2^{+2} + a_2^2] \qquad J_{35} = \tfrac{1}{2}\mathrm{i}[a_1^+a_2^+ - a_1a_2] \tag{2.4}$$

$$J_{34} = -\tfrac{1}{2}[a_1^+a_2^+ + a_1a_2] \qquad J_{45} = \tfrac{1}{2}[a_1^+a_1 + a_2^+a_2 + 1] \tag{2.5}$$

that satisfy the conmutation rules:

$$[J_{ab}, J_{cd}] = \mathrm{i}[g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac}] \tag{2.6}$$

with $g_{ij} = 0$ if $i \neq j$; $g_{ij} = 1$ if $i = 1, 2, 3$ and $g_{ij} = -1$ if $i = 4, 5$ and yield to determine the set of the operators J_{ab} ($a = 1, 2, 3, 4$ and $b = 2, 3, 4, 5$) as a realization of the Lie algebra $so(3, 2) \approx sp(4, R)$. The dynamical symmetry of the Hamiltonian (1.1) is established in such a way.

By considering the linear combinations:

$$C_0 = J_{12} = \tfrac{1}{2}[a_1^+ a_1 - a_2^+ a_2] \qquad D_0 = J_{45} = \tfrac{1}{2}[a_1^+ a_1 + a_2^+ a_2 + 1] \tag{2.7}$$

$$C_+ = J_{13} - \mathrm{i}J_{23} = a_1^+ a_2 \qquad D_+ = -J_{34} - \mathrm{i}J_{35} = a_1^+ a_2^+ \tag{2.8}$$

$$C_- = J_{13} + \mathrm{i}J_{23} = a_1 a_2^+ \qquad D_- = -J_{34} + \mathrm{i}J_{35} = a_1 a_2 \tag{2.9}$$

and

$$E_0 = \tfrac{1}{2}(J_{12} + J_{45}) = \tfrac{1}{2}\left[a_1^+ a_1 + \tfrac{1}{2}\right] \tag{2.10}$$

$$E_+ = \tfrac{1}{2}(J_{15} - J_{24}) - \tfrac{1}{2}\mathrm{i}(J_{14} + J_{25}) = \tfrac{1}{2}a_1^{+2} \tag{2.11}$$

$$E_- = \tfrac{1}{2}(J_{15} - J_{24}) + \tfrac{1}{2}\mathrm{i}(J_{14} + J_{25}) = \tfrac{1}{2}a_1^2 \tag{2.12}$$

$$F_0 = \tfrac{1}{2}(J_{45} - J_{12}) = \tfrac{1}{2}\left[a_2^+ a_2 + \tfrac{1}{2}\right] \tag{2.13}$$

$$F_+ = \tfrac{1}{2}(J_{15} + J_{24}) - \tfrac{1}{2}\mathrm{i}(J_{14} - J_{25}) = \tfrac{1}{2}a_2^{+2} \tag{2.14}$$

$$F_- = \tfrac{1}{2}(J_{15} + J_{24}) + \tfrac{1}{2}\mathrm{i}(J_{14} - J_{25}) = \tfrac{1}{2}a_2^2 \tag{2.15}$$

as well as their conmutation relations, we can obtain for $so(3, 2)$ the subalgebra structure given in table 1.

Table 1. Subalgebras of $so(3, 2)$.

Generators	Subalgebra	Casimir
$C = \{C_0, C_+, C_-\}$	$so(3) \approx su(2)$	$C^2 = C_0^2 + \frac{1}{2}(C_+ C_- + C_- C_+)$
$D = \{D_0, D_+, D_-\}$	$so(2, 1) \approx su(1, 1)$	$D^2 = -D_0^2 + \frac{1}{2}(D_+ D_- + D_- D_+)$
$E = \{E_0, E_+, E_-\}$	$so(2, 1) \approx su(1, 1)$	$E^2 = -E_0^2 + \frac{1}{2}(E_+ E_- + E_- E_+)$
$F = \{F_0, F_+, F_-\}$	$so(2, 1) \approx su(1, 1)$	$F^2 = -F_0^2 + \frac{1}{2}(F_+ F_- + F_- G_+)$
$E \cup F$	$su(1, 1) + su(1, 1)$	
$\{C_0\} \cup D$	$u(1) + su(1, 1)$	
$\{D_0\} \cup C$	$u(1) + su(2)$	

The two independent Casimir operators [7] in the realizations are

$$C_1 = 2E_0 + 2E^2 + 2F^2 + D_+ D_- - C_+ C_- = \tfrac{5}{4} \tag{2.16}$$

$$C_2 = \tfrac{1}{2}\left\{[C_+, \Phi]^2 + [C_-, \Phi]^2 - [D_+, \Phi]^2 - [D_+, \Phi]^2 - 2\Phi^2\right\} = 0 \tag{2.17}$$

with $\Phi = E^2 - F^2$.

The Hilbert space of $H(t)$ is the space of the representation $\left(\frac{5}{4}, 0\right)$ of $sp(4, R)$. Such a representation contains the direct product of the representations $D^+(E)^* D^+(F)$ of $su(1, 1) + su(1, 1)$ with E, F, $\frac{1}{4}$ or $\frac{3}{4}$, the states of which have a definite number of photons. The Hilbert space, in this realization of the algebra, should therefore be generated by states with a definite number of photons in each mode. Notice that the compact maximal subalgebra, $u(1) + su(2)$, is constructed with the operators that conserve the total number of photons.

3. Group elements and the temporal evolution

By using the exponential mapping we can express each group element as

$$\exp\left\{\sum_{ij}\alpha_{ij}J_{ij}\right\} \qquad i,j:1\ldots 5. \tag{3.1}$$

If we consider just unitary elements, we can find, after a suitable factorization, elements as [8]

$$U = S_1 S_2 S_{12} T R_1 R_2 \tag{3.2}$$

where the notation is quite straightforward:

$$S_1 = S_1(\beta_1, a_1) = \exp\left[\tfrac{1}{2}\left(\beta_1 a_1^{+2} - \beta_1^* a_1^2\right)\right] \tag{3.3}$$

$$S_2 = S_2(\beta_2, a_2) = \exp\left[\tfrac{1}{2}\left(\beta_2 a_2^{+2} - \beta_2^* a_2^2\right)\right] \tag{3.4}$$

$$S_{12} = S_{12}(\beta, a_1, a_2) = \exp[\beta a_1^+ a_2^+ - \beta^* a_1 a_2] \tag{3.5}$$

$$T = T(\tau, a_1, a_2) = \exp[\tau a_1^+ a_2 - \tau^* a_1 a_2^+] \tag{3.6}$$

$$R_1 = R_1(\theta_1, a_1) = \exp\left[\tfrac{1}{2}\mathrm{i}\theta_1\left(a_1^+ a_1 + \tfrac{1}{2}\right)\right] \tag{3.7}$$

$$R_2 = R_2(\theta_2, a_2) = \exp\left[\tfrac{1}{2}\mathrm{i}\theta_2\left(a_2^+ a_2 + \tfrac{1}{2}\right)\right]. \tag{3.8}$$

The Baker–Hausdorff–Campbell formula allows us to rewrite them in a more useful way that is summarized in table 2; notice that the proposed factorization is always possible for the symmetry $SU(1,1)$ but not for $SU(2)$. The reason lies in the different behaviour of the function $\tanh(x)$ which is continuous in the positive halfline and $\tan(x)$ which has discontinuities at $x = (2l+1)\frac{\pi}{2}$. For instance, it is not possible to split the operator $T\left(\frac{1}{2}\pi \mathrm{e}^{\mathrm{i}\Phi}, a_1, a_2\right)$. Therefore we accept $0 \leqslant q < \pi/2$ (see table 2).

Table 2. Factorization of the operators.

$S_i(\beta_i, a_i) = S_i(\eta_i, a_i) = \exp\{(\eta_i/2)\}a_i^{+2}\}\exp\{(\gamma_i/2)(a_i^+ a_i + 1/2)\}\exp\{-(\eta_i/2)a_i^2\}$		
$S_{12}(\beta, a_1, a_2) = S_{12}(\eta, a_1, a_2) = \exp\{\eta a_1^+ a_2^+\}\exp\{(\gamma/2)(a_1^+ a_1 + a_2^+ a_2 + 1)\}\exp\{-\eta^* a_1 a_2\}$		
$T(\tau, a_1, a_2) = T(\mu, a_1, a_2) = \exp\{\mu a_1^+ a_2\}\exp\{(\pi/2)(a_1^+ a_1 - a_2^+ a_2)\}\exp\{-\mu^* a_1 a_2^+\}$		
$\beta_i = r_i \exp\{\mathrm{i}\phi_i\}$	$\beta = r\exp\{\mathrm{i}\phi\}$	$\tau = q\exp\{\mathrm{i}\Phi\}$
$\eta_i = \tanh r_i \exp\{\mathrm{i}\phi_i\}$	$\eta = \tanh r \exp\{\mathrm{i}\phi\}$	$\mu = \tan q \exp\{\mathrm{i}\Phi\}$
$\gamma_i = \log\{1 - \eta_i \eta_i^*\}$	$\gamma = \log\{1 - \eta\eta^*\}$	$\pi = \log\{1 + \mu\mu^*\}$

We shall be calling $S_i(\beta_i, a_i)$ the $SU(1,1)$ generalized coherent state generator as constructed in [9–11]. Its action on the creation and annihilation operators

$$S_j a_j S_j^+ = \frac{a_j - \eta_j a_j^+}{\sqrt{1 - |\eta_j|^2}} = \cosh r_j a_j - \mathrm{e}^{\mathrm{i}\Phi_j}\sinh r_j a_j^+ \tag{3.9}$$

transforms them into a linear combination of the previous canonical operators, which has the effect of transforming an initial generalized coherent state into a one-photon coherent mixture with different noise properties as is shown in table 3—we call it the *one-mode* squeezing operator.

Table 3. Bogolyubov transformations.

$S_j a_i S_j^+ = (a_j - \eta_j a_j^+)(1 - \|\eta_j\|^2)^{-1/2} = \cosh r_j a_j - e^{i\phi_j} \sinh r_j a_j^+$
$S_i a_j S_i^+ = a_j \qquad i \neq j$
$S_{12} a_1 S_{12}^+ = (a_1 - \eta a_2^+)(1 - \|\eta\|^2)^{-1/2} = \cosh r a_1 - e^{i\phi} \sinh r a_2^+$
$S_{12} a_2 S_{12}^+ = (a_2 - \eta a_1^+)(1 - \|\eta\|^2)^{-1/2} = \cosh r a_2 - e^{i\phi} \sinh r a_1^+$
$T a_1 T^+ = (a_1 - \mu a_2)(1 + \|\mu\|^2)^{-1/2} = \cos q a_1 - e^{i\Phi} \sin q a_2$
$T a_2 T^+ = (a_2 + \mu^* a_1)(1 + \|\mu\|^2)^{-1/2} = \cos q a_2 + e^{-i\Phi} \sin q a_1$
$R_j a_j R_j^+ = \exp\{-\frac{1}{2} i\theta_j\} a_j$
$R_i a_j R_i^+ = a_j \qquad i \neq j$

$S(\beta, a_1, a_2)$ is the two-mode $SU(1,1)$ generalized coherent state generator, which leads to the two-mode squeezing operator

$$S_{12} a_1 S_{12}^+ = \frac{a_1 - \eta a_2^+}{\sqrt{1 - |\eta|^2}} = \cosh r a_1 - e^{i\phi} \sinh r a_2^+ \tag{3.10}$$

$$S_{12} a_2 S_{12}^+ = \frac{a_2 - \eta a_1^+}{\sqrt{1 - |\eta|^2}} = \cosh r a_2 - e^{i\phi} \sinh r a_1^+ \,. \tag{3.11}$$

These transformations mix the linear combinations of the canonical operators of both modes. The noise properties for the transformed operators also appear to be drastically modified.

$T(\tau, a_1, a_2)$ is the compact operator belonging to the set of $SU(2)$ generalized coherent state generators. This operator, as will be shown below, does not modify the noise properties of the transformed operator. Its action can be written as

$$T a_1 T^+ = \frac{a_1 - \mu a_2}{\sqrt{1 + |\mu|^2}} = \cos q a_1 - e^{i\Phi} \sin q a_2 \tag{3.12}$$

$$T a_2 T^+ = \frac{a_2 + \mu^* a_1}{\sqrt{1 + |\mu|^2}} = \cos q a_2 + e^{-i\Phi} \sin q a_1 \,. \tag{3.13}$$

$R_i(\theta_i, a_i)$ is merely a rotation operator in the complex plane acting as

$$R_j a_j R_j^+ = \exp\left\{-\tfrac{1}{2} i\theta_j\right\} a_j \tag{3.14}$$

$$R_i a_j R_i^+ = a_j \qquad i \neq j \tag{3.15}$$

and it transforms a given generalized coherent state into another state with an eigenvalue rotated by an angle θ_i. Indeed it does not modify the noise properties. A brief account of these properties is given in table 4. The temporal evolution of the significant elements can be obtained from the Magnus formula [12] and is summarized in table 5.

Table 4. Generators of coherent states.

Generators	Subgroup
$T = T(\tau, a_1, a_2)$	$SU(2)$
$S_1 = S_1(\beta_1, a_1)$	$SU(1,1)$
$S_2 = S_2(\beta_2, a_2)$	$SU(1,1)$
$S_{12} = S_{12}(\beta, a_1, a_2)$	$SU(1,1)$

Table 5. Temporal evolution of the operators.

$\dot{S}_i S_i^+ = \frac{\dot{\eta}_i}{1-\vert\eta_i\vert^2} \frac{a_i^{+2}}{2} - \frac{1}{2} \frac{\dot{\eta}_i \eta_i^* - \eta_i \dot{\eta}_i^*}{1-\vert\eta_i\vert^2} (a_i^+ a_i + \frac{1}{2}) - \frac{\dot{\eta}_i^*}{1-\vert\eta_i\vert^2} \frac{a_i^2}{2}$
$\dot{S}_{12} S_{12}^+ = \frac{\dot{\eta}}{1-\vert\eta\vert^2} a_1^+ a_2^+ - \frac{1}{2} \frac{\dot{\eta}\eta^* - \eta\dot{\eta}^*}{1-\vert\eta\vert^2} (a_1^+ a_1 + a_2^+ a_2 + 1) - \frac{\dot{\eta}^*}{1-\vert\eta\vert^2} a_1 a_2$
$\dot{T} T^+ = \frac{\dot{\mu}}{1+\vert\mu\vert^2} a_1^+ a_2 + \frac{1}{2} \frac{\dot{\mu}\mu^* - \mu\dot{\mu}^*}{1+\vert\mu\vert^2} (a_1^+ a_1 - a_2^+ a_2) - \frac{\dot{\mu}^*}{1+\vert\mu\vert^2} a_1 a_2^+$
$\dot{R}_j R_j^+ = \mathrm{i} \frac{\dot{\theta}_j}{2} (a_j^+ a_j + \frac{1}{2})$

4. Temporal evolution and diagonalization

The exact solution of the Schrödinger equation for the Hamiltonian (1.1) is a problem of great difficulty that relies on the existence of two degrees of freedom with a strong nonlinear interaction which contains the 10-parameter dynamical group symmetry $Sp(4, R)$. To solve the problem it is necessary to deal with a nonlinear system of 10 coupled differential equations. Nevertheless, in some particular cases, with interesting physical applications, the system has exact solutions. These particular cases are found by reducing the system to one effective degree of freedom. The reduction can be obtained either through a complete and systematic classification of the Lie subalgebras of the system or by specifying the parameters of the Hamiltonian and its corresponding relationship to them. Both methods can be shown to be totally equivalent. Throughout this paper we have used the second method.

The temporal evolution operator can be written as an element of one of the above-mentioned subgroups in such a way that the whole problem can be solved just by using the subgroup elements, and the elements of the total group are, in principle, not necessary. The instantaneous diagonalization will be treated in a similar manner: the triparametric reductions of $H(t)$ are related through a unitary operator constructed with the help of the corresponding Cartan subalgebra of each subgroup (see table 4).

4.1. Reduction to $su(2) + u(1)$, $g_1 = g_2 = d = 0$

$$H(t) = 2s_1\left(a_1^+ a_1 + \tfrac{1}{2}\right) + 2s_2\left(a_2^+ a_2 + \tfrac{1}{2}\right) + 2e a_1^+ a_2 + 2e^* a_1 a_2^+ \tag{4.1}$$

where $H(t)$ could be interpreted as the Hamiltonian describing the ideal conversion of one photon of frequency $\omega_2 < \omega_1$ and another one of frequency $\omega_1 - \omega_2$ (coming from a classical pumping field) which are simultaneously destroyed to produce a single photon of frequency ω_1. The coherent pumping light beam is treated as an unlimited source of photons described by the classical function $e(t)$.

$H(t)$ is also an element of the algebra $so(3, 2)$ and its subalgebra $su(2) + u(1)$. The states belonging to this representation space can be classified by the number of photons in each mode. The relationship between these photon numbers and the eigenvalues of the $su(2)$ representation, j and m, are given by

$$j = \tfrac{1}{2}(k_1 + k_2) \tag{4.2}$$

$$m = \tfrac{1}{2}(k_1 - k_2)\,. \tag{4.3}$$

It is trivial to see that all possible $su(2)$ representations may be realized with k_1 and k_2 as arbitrary integers.

The group elements and, in particular, the time evolution operator act in an irreducible manner on the representation space vector states for a fixed value of j. In this way they leave invariant the total number of photons but not the individual number in each mode.

4.1.1. Instantaneous diagonalization. One can always achieve an instantaneous diagonalization even if the dynamical group is not $SU(2)$ (as is the case here). The unitary operator (an element of the group) acting on the Cartan subalgebra will now be a mixing operator and $H(t)$ can be expressed in the instantaneous approximation as

$$H(t) = T(\mu_0)\{2(s_1 + s_2)D_0 + 2F(t)C_0\}T^+(\mu_0) \tag{4.4}$$

for certain values of the functions $F(t)$ and $\mu_0(t)$ in connection with the Hamiltonian functions. For instance,

$$F(t) = \sqrt{(s_1 - s_2)^2 + 4|e|^2}\,. \tag{4.5}$$

The instantaneous eigenstates of $H(t)$ are the generalized coherent states $T(\mu_0(t), a_1, a_2)|k_1, k_2\rangle$ corresponding to a realization in two modes of $SU(2)$ with a time-dependent eigenvalue

$$E_{k_1,k_2} = (s_1 + s_2)(k_1 + k_2 + 1) + F(t)(k_1 - k_2)\,. \tag{4.6}$$

Notice that $F(t)$ is real for any value of the parameters. This means that any $SU(2)$ operator is diagonalizable and (in particular for this realization) any two-dimensional oscillator can be put in correspondence through unitary operators to two independent stationary harmonic oscillators.

4.1.2. Temporal evolution. The direct product implies that the evolution operator can be constructed by means of the product of two unitary elements, one from $U(1)$ and the other from $SU(2)$

$$U(t) = T(\tau, a_1, a_2)R_1(\theta_1, a_1)R_2(\theta_2, a_2) \tag{4.7}$$

determined by the characteristic functions defined in terms of the parameters of the Hamiltonian by means of the equations

$$\theta_1 = -\frac{2}{\hbar}\int_0^t (2s_1 - e^*\mu - e\mu^*)\,\mathrm{d}t \tag{4.8}$$

$$\theta_2 = -\frac{2}{\hbar}\int_0^t (2s_2 + e^*\mu + e\mu^*)\,\mathrm{d}t \tag{4.9}$$

$$\dot{\mu} = -\mathrm{i}\frac{2}{\hbar}[e + (s_1 - s_2)\mu - e^*\mu^2] \qquad \mu(0) = 0\,. \tag{4.10}$$

4.2. Reduction to $su(1,1) + U(1)$, $g_1 = g_2 = e = 0$

$$H(t) = 2s_1\left(a_1^+a_1 + \tfrac{1}{2}\right) + 2s_2\left(a_2^+a_2 + \tfrac{1}{2}\right) + 2da_1^+a_2^+ + 2d^*a_1a_2 \tag{4.11}$$

where $H(t)$ should now be regarded as describing the ideal frequency conversion in which a photon of frequency $\omega_1 + \omega_2$ of a laser beam in a classical coherent state is destroyed when interacting with a material media. The result is the creation of two photons with frequencies ω_1 and ω_2; this process takes place in the degenerate optical parametric operators [13–17].

$H(t)$ is now an element of the $su(1,1) + u(1)$ part of the $sp(4, R)$ algebra. The realization of $su(1,1)$ that we should consider here uses two bosonic operators that describe two independent vibration modes and differs from that used in the one degree of freedom case [1]. The eigenvalue of the Casimir operator is

$$D^2 = \tfrac{1}{4} - C_0^2 \tag{4.12}$$

and is not a fixed value but depends on the occupation number of both modes as well as on the eigenvalue C_0. Equation (4.12) has two solutions: to each state $|k_1, k_2\rangle$ with a given occupation degree corresponds two different values of (k, m). One can freely choose one of them. The states $|k_1, k_2\rangle = |k, k+m\rangle$ are identified with

$$k = \frac{k_1 - k_2 + 1}{2} \qquad (4.13)$$

$$m = k_2 \, . \qquad (4.14)$$

In particular, all the states with equal numbers of photons in both modes constitute the shell $k = 0, m = 0, 1, 2$, etc of $SU(1,1)$.

The elements of the group now leave invariant the *difference* of the number of photons in each mode but not the total number. Therefore the action of the group elements on the space of states is equivalent to creating (or annihilating) pairs of photons of each mode simultaneously.

4.2.1. Instantaneous diagonalization. Calling ϵ the sign of $s_1 + s_2$, $H(t)$ can be expressed as

$$H(t) = S_{12}(\eta_0)\{2(s_1 - s_2)C_0 + 2\epsilon F(t) D_0\} S_{12}^{+}(\eta_0) \qquad (4.15)$$

for certain values of $F(t)$ and $\eta_0(t)$ independent of the functions of the Hamiltonian. In particular,

$$F(t) = \sqrt{(s_1 + s_2)^2 - 4|d|^2} \qquad (4.16)$$

$$\eta_0(t) = -\epsilon \frac{2d(t)}{|s_1 + s_2| + F} \, . \qquad (4.17)$$

The instantaneous eigenstates of $H(t)$ are the generalized coherent states $S[\eta_0(t), a_1, a_2]|k_1, k_2\rangle$ corresponding to a two-mode realization of $SU(1,1)$ with a time-dependent eigenvalue

$$E_{k_1,k_2}(t) = (s_1 - s_2)(k_1 - k_2) + \epsilon F(t)(k_1 + k_2 + 1) \, . \qquad (4.18)$$

This method can only be applied if the following condition holds:

$$(s_1 + s_2)^2 - 4|d|^2 > 0 \, . \qquad (4.19)$$

This is due to the fact that $F(t)$ is not well defined if this condition is not fulfilled.

4.2.2. Temporal evolution. In this case the temporal evolution operator should be factorized as the product of a two-mode 'squeezing' operator by a rotation operator

$$U(t) = S_{12}(\beta, a_1, a_2) R_1(\theta_1, a_1) R_2(\theta_2, a_2) \qquad (4.20)$$

whose characteristic functions are

$$\theta_1 = -\frac{2}{\hbar}\int_0^t (2s_2 + d^*\eta + d\eta^*)\,\mathrm{d}t \qquad (4.21)$$

$$\theta_2 = -\frac{2}{\hbar}\int_0^t (2s_1 + d^*\eta + d\eta^*)\,\mathrm{d}t \qquad (4.22)$$

$$\dot{\eta} = -\mathrm{i}\frac{2}{\hbar}[d + (s_1 + s_2)\eta + d^*\eta^2] \qquad \eta(0) = 0 \, . \qquad (4.23)$$

The exact states of $H(t)$ can be obtained with the aid of table 2 as

$$|\Psi(t)\rangle = S_{12}(\beta, a_1, a_2) R_1(\theta_1, a_1) R_2(\theta_2, a_2)|k_1, k_2\rangle \, . \qquad (4.24)$$

4.3. *Reduction to* $su(1,1) + su(1,1)$, $d = e = 0$

The Hamiltonian is

$$H = \sum_{i=1}^{2} 2s_i\left(a_i^+ a_i + \tfrac{1}{2}\right) + g_i a_i^{+2} + g_i^* a_i^2 \,. \tag{4.25}$$

The initial Hamiltonian is an element of the part $su(1,1) + su(1,1)$ of the $so(3,2)$ algebra. Because of the conmutation of all the operators of the one-mode with those of the two-mode, the problem can be factorized and its solution requires the solution of two independent identical systems.

4.3.1. Instantaneous diagonalization. The operators $S = S_1(\beta_1, a_1)S_2(\beta_2, a_2)$ (see table 2) diagonalize $H(t)$ exactly for all times [1]. In this case the relevant parameters take the form

$$H(t) = \sum_{i=1}^{2} 2\epsilon_i F_i(t) S_i\,(\eta_{0i})\left(a_i^+ a_i + \tfrac{1}{2}\right) S_i^+\,(\eta_{0i}) \qquad i = 1, 2 \tag{4.26}$$

$$F_i(t) = \sqrt{s_i^2 - |g_i|^2} \qquad i = 1, 2 \tag{4.27}$$

$$\eta_{0i}(t) = -\epsilon_i \frac{g_i}{|s_i| + F_i} \qquad i = 1, 2\,. \tag{4.28}$$

4.3.2. Temporal evolution. The evolution operator is $U(t) = U_1(t)U_2(t)$ with $U_i(t)$ defined as

$$U_i(t) = S_i(\beta_i, a_i) R_i(\theta_i, a_i) \tag{4.29}$$

the characteristic functions of which are [1]

$$\theta_i = -\frac{2}{\hbar}\int_0^t (2s_i + g_i^*\eta_i + g_i\eta_i^*)\,\mathrm{d}t \qquad i = 1, 2 \tag{4.30}$$

$$\dot{\eta}_i = -\frac{2\mathrm{i}}{\hbar}(g_i + 2s_i\eta_i + g_i^*\eta_i^2)\,\mathrm{d}t \qquad \eta_i(0) = 0 \qquad i = 1, 2\,. \tag{4.31}$$

4.4. $g_1 = g_2 = s_1 = s_2 = 0$

$$H = 2d^* a_1 a_2 + 2e a_1^+ a_2 + \text{HC}\,. \tag{4.32}$$

The initial Hamiltonian does not belong to the subalgebras considered and contains just terms of the interaction between the two modes. In some cases the two modes in the interaction should be decoupled by means of a canonical transformation and the problem could be solved in terms of two independent particles. In fact, we can introduce a new pair of modes

$$\bar{a}_1 = X a_1 X^+ = \sqrt{\frac{e}{|e|}}\,\frac{a_1 - a_2}{\sqrt{2}} \tag{4.33}$$

$$\bar{a}_2 = X a_2 X^+ = \sqrt{\frac{e^*}{|e|}}\,\frac{a_1 + a_2}{\sqrt{2}} \tag{4.34}$$

that are canonically related to the initial modes through the product of one mixed operator by a rotation operator,

$$X = T\left(\frac{\pi}{4}, a_1, a_2\right) R_1(-\arg(e), a_1) R_2(\arg(e), a_2)\,. \tag{4.35}$$

The time-dependent system transformed by X is

$$\bar{H} = \mathrm{i}\hbar\dot{X}X^{+} + XHX^{+} = -\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\arg(e)]a_1^{+}a_2 + d^{*}(a_1^2 - a_2^2) + |e|(a_1^{+}a_1 - a_2^{+}a_2) + \mathrm{HC}\,. \tag{4.36}$$

The only time dependence in X is the one coming from $\arg(e)$ in such a way that if this argument is constant (in particular, if $e(t)$ is real or purely imaginary), the canonically transformed Hamiltonian is an element of the $su(1,1) + su(1,1)$ subalgebra that corresponds to two independent particles. The system is reduced to the case discussed in subsection 4.3. In any case, the same transformation X could be used to obtain the instantaneous diagonalization of the Hamiltonian.

4.4.1. Instantaneous diagonalization. The term XHX^{+} actually represents two uncoupled one-dimensional harmonic oscillators. For a given time t, the Hamiltonian can actually be written as

$$H(t) = F(t)X^{+}S_1(\eta_0, a_1)S_2(\eta_0, a_2)\{a_1^{+}a_1 - a_2^{+}a_2\}S_1^{+}(\eta_0, a_1)S_2^{+}(\eta_0, a_2)X \tag{4.37}$$

for certain values of $F(t)$ and $\eta_0(t)$ as functions of the parameters in the form

$$F(t) = \sqrt{|e|^2 - 4|d|^2} \tag{4.38}$$

$$\eta_0(t) = -\frac{2d}{F(t) + |e|}\,. \tag{4.39}$$

The instantaneous eigenstate of $H(t)$ are the states given by

$$X^{+}S_1(\eta_0, a_1)S_2(\eta_0, a_2)|k_1, k_2\rangle \tag{4.40}$$

with time-dependent eigenvalues

$$E_{k_1k_2}(t) = F(t)(k_1 - k_2)\,. \tag{4.41}$$

This diagonalization can only be made possible if the condition $|e| > 2|d|$ holds, and the set of eigenstates becomes highly degenerate. All states with the same difference $k_1 - k_2$ have the same eigenvalue.

4.4.2. Temporal evolution. The state corresponding to $\bar{H}$ can easily be found by the evolution of $H(t)$ using $\bar{U}(t) = \bar{U}_1(t)\bar{U}_2(t)$ given by

$$\bar{U}_i(t) = S_i(\beta_i, a_i)R_i(\theta_i, a_i) \qquad i = 1, 2 \tag{4.42}$$

and the corresponding functions are

$$\theta_i = \pm\frac{2}{\hbar}\int_0^t (2|e| + d^{*}\eta_i + d\eta_i^{*})\,\mathrm{d}t \qquad i = 1, 2 \tag{4.43}$$

$$\dot{\eta}_i = \pm\frac{2}{\mathrm{i}\hbar}[d + 2|e|\eta_i + d^{*}\eta_i^2] \qquad \eta_i(0) = 0 \qquad i = 1, 2\,. \tag{4.44}$$

As soon as $\bar{U}(t)$ has been explicitly obtained one can use it to find $U(t) = X^{+}\bar{U}(t)X$. We have been able to demonstrate that using the conmutation relations:

$$R_1(\theta_1)R_2(\theta_2)T(\tau) = T\left(\tau \mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)/2}\right)R_1(\theta_1)R_2(\theta_2) \tag{4.45}$$

$$R_i(\theta_i)S_i(\eta_i) = S_i\left(\eta_i \mathrm{e}^{\mathrm{i}\theta_i}\right)R_i(\theta_i)\,. \tag{4.46}$$

The unitary operator $U(t)$ can be factorized as

$$U(t) = T\left(-\frac{\pi}{4}\delta_e\right)S_1(\eta_1\delta_e)S_2(\eta_2\delta_e^{*})R_1(\theta_1)R_2(\theta_2)T\left(\frac{\pi}{4}\delta_e\right) \tag{4.47}$$

with $\delta_e = \mathrm{e}^{\mathrm{i}\arg(e)}$.

An alternative way to deal with the complete case relies upon the use of the interaction picture. Let a Hamiltonian H_0, $(su(1,1) + su(1,1))$-invariant be of the form

$$H_0 = \sum_{j=1}^{2} \left[2s_j\left(a_j^+ a_j + \tfrac{1}{2}\right) + g_j a_j^{+2} + g_j^* a_j^2\right] \tag{4.48}$$

with time evolution operator given by $U_0(t)$ of section 4.3 and the interaction Hamiltonian is given by

$$\begin{aligned} H_{int} &= U_0^+[2da_1^+a_2^+ + 2d^*a_1a_2 + 2ea_1^+a_2 + 2e^*a_1a_2^+]U_0 \\ &= 2\bar{d}a_1^+a_2^+ + 2\bar{d}^*a_1a_2 + 2ea_1^+a_2 + 2\bar{e}a_1^+a_2 + 2\bar{e}^*a_1a_2^+ \end{aligned} \tag{4.49}$$

where

$$\bar{d} = \frac{\delta_1^*\delta_2^*}{\sqrt{1-|\eta_1|^2}\sqrt{1-|\eta_2|^2}}[d + d^*\eta_1\eta_2 + e^*\eta_1 + e\eta_2] \tag{4.50}$$

$$\bar{e} = \frac{\delta_1^*\delta_2}{\sqrt{1-|\eta_1|^2}\sqrt{1-|\eta_2|^2}}[d^*\eta_1 + d\eta_2^* + e + e^*\eta_1\eta_2^*] \tag{4.51}$$

with $\delta_j = \mathrm{e}^{\mathrm{i}\theta_j/2}$, $j = 1, 2$.

If one should be able to find the exact evolution operator $U_{int}(t)$ for H_{int}, the problem would have been easily solved since the time evolution of the initial system will be given by $U(t) = U_0(t)U_{int}(t)$. Therefore the problem can be reduced to solve the case of section 4.4 where $\arg(e)$ is not a constant. We have been able to establish after a long calculation that if one constraint holds this goal can be achieved. The constraint is†

$$\arg(\bar{e}) = \frac{\theta_2 - \theta_1}{2} + \arg\tan(u) = cte \tag{4.52}$$

where u is given by the following expression:

$$u = \frac{|d|[|\eta_1|\sin\Xi_1 + |\eta_2|\sin\Xi_2] + |e|[\sin(\arg e) + |\eta_1||\eta_2|\sin(\Xi_1 - \Xi_2 - \arg e)]}{|d|[|\eta_1|\cos\Xi_1 + |\eta_2|\cos\Xi_2] + |e|[\cos(\arg e) + |\eta_1||\eta_2|\cos(\Xi_1 - \Xi_2 - \arg e)]} \tag{4.53}$$

and

$$\Xi_i = \arg\eta_i - \arg d\,. \tag{4.54}$$

To this class of general solutions belongs the Hamiltonian solved in [2] with the identifications

$$e(t) = \hbar\Omega \tag{4.55}$$

$$g_1 = -g_2 = -\mathrm{i}\frac{\hbar}{2}\Gamma \tag{4.56}$$

$$s = d = 0\,. \tag{4.57}$$

In our formalism we shall be obtaining

$$\eta_2 = -\eta_1 = \tanh\left\{\int_0^t \Gamma(s)\,\mathrm{d}s\right\} \tag{4.58}$$

$$\theta_2 = \theta_1 = \bar{d} = 0 \qquad \bar{e} = e \tag{4.59}$$

$$H_{int} = 2\hbar\Omega(a_1^+a_2 + a_1a_2^+) \tag{4.60}$$

which is clearly $SU(2)$-invariant.

† Notice that the constraint involves a relationship of 10 functions. However restrictive it might seem there is still quite a lot of room for many interesting cases that can be solved exactly as we shall show below.

4.5. $g_1 = g, g_2 = g\mathrm{e}^{-2\mathrm{i}\arg(e)}, s_1 = s_2 = s, \arg(e) = cte$

The Hamiltonian is

$$H = g^*\left(a_1^2 + \mathrm{e}^{2\mathrm{i}\arg(e)}a_2^2\right) + s(a_1^+a_1 + a_2^+a_2 + 1) + 2d^*a_1a_2 + 2ea_1^*a_2 + \mathrm{HC}\,. \tag{4.61}$$

Besides the fact that the condition among the coefficients is strong, in fact it contains the important cases: (i) $g_1 = g_2$ and $e(t)$ real (symmetry of interchange), (ii) $g_1 = -g_2$ and $e(t)$ imaginary and (iii) $g_1 = g_2 = 0,\ s_1 = s_2$ and $e(t)$ imaginary which corresponds to the problem discussed in [18] for the degenerate case. This case is technically identical to the previous one. The same canonical transformation $X(t)$ drives the system to a set of two particles whose interaction is determined for the time evolution of $\arg(e)$,

$$\begin{aligned}\bar{H} &= \mathrm{i}\hbar\dot{X}X^+ + XHX^+ \\ &= -\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\arg(e)]a_1^+a_2 + (g\mathrm{e}^{-\mathrm{i}\arg(e)} + d)a_1^{+2} + (g\mathrm{e}^{-\mathrm{i}\arg(e)} - d)a_2^{+2} \\ &\quad + (s + |e|)\left(a_1^+a_1 + \tfrac{1}{2}\right) + (s - |e|)\left(a_2^+a_2 + \tfrac{1}{2}\right) + \mathrm{HC}\,.\end{aligned} \tag{4.62}$$

4.5.1. Instantaneous diagonalization. The static part of $\bar{H}$ of sections 4.4 and 4.5 represent the same and differ just by their characteristic parameters. The fact that the problem has been reduced to two one-dimensional independent harmonic generalized oscillators with different functions forces us to consider transformations with different parameters $\eta_0^{(1)}$ and $\eta_0^{(2)}$. If we call $\epsilon_\pm$ the sign of $(s \pm |e|)$:

$$H(t) = \Delta\{\epsilon_+F_1(t)a_1^+a_1 + \epsilon_-F_2(t)a_2^+a_2\}\Delta^+ \tag{4.63}$$

where

$$\Delta = X^+S_1(\eta_0^{(1)}, a_1)S_2(\eta_0^{(2)}, a_2) \tag{4.64}$$

with

$$F_{1,2}(t) = \sqrt{(s \pm |e|)^2 - 4(|d|^2 + |g|^2 \pm 2|d||g|\cos(\varphi_g - \varphi_d - \varphi_e))} \tag{4.65}$$

$$\eta_0^{(1,2)}(t) = -\epsilon_\pm\frac{2(g\mathrm{e}^{-\mathrm{i}\varphi_e} \pm d)}{|(s \pm |e|)| + F_{1,2}}\,. \tag{4.66}$$

The instantaneous eigenstates of $H(t)$ are

$$X^+S_1(\eta_0^{(1)}, a_1)S_2(\eta_0^{(2)}, a_2)|k_1, k_2\rangle \tag{4.67}$$

whose time-dependent eigenvalue is

$$E_{k_1,k_2}(t) = \epsilon_+F_1(t)k_1 + \epsilon_-F_2(t)k_2\,. \tag{4.68}$$

4.5.2. Temporal evolution. If $\arg(e)$ is a constant, we are again in the case of section 4.3. The solution can be written as in (4.26) where

$$\theta_j = -\frac{2}{\hbar}\int_0^t\left[2(s \pm |e|) + (g^*\mathrm{e}^{\mathrm{i}\arg(e)} \pm d^*)\eta_j + (g\mathrm{e}^{-\mathrm{i}\arg(e)} \pm d)\eta_j^*\right]\mathrm{d}t \tag{4.69}$$

$$\dot{\eta}_j = \frac{2}{\mathrm{i}\hbar}\left[(g\mathrm{e}^{-\mathrm{i}\arg(e)} \pm d) + 2(s \pm |e|)\eta_j + (g^+\mathrm{e}^{\mathrm{i}\arg(e)} \pm d^*)\eta_j^2\right] \tag{4.70}$$

$$\eta_j(0) = 0 \qquad j = 1, 2\,. \tag{4.71}$$

4.6. The general case

Now let $H(t)$ be a general Hamiltonian described by

$$H(t) = s_1(t)\left(a_1^+ a_1 + \tfrac{1}{2}\right) + s_2(t)\left(a_2^+ a_2 + \tfrac{1}{2}\right) + g_1(t)a_1^{+2} + g_2(t)a_2^{+2} + 2d(t)a_1^+ a_2^+ + 2e(t)a_1^+ a_2 + \mathrm{HC} \tag{4.72}$$

where a_1, a_1^+, a_2, a_2^+ are canonical operators associated with two different modes. Let us find the most general solution of a time evolution operator $U(t)$ verifying

$$\mathrm{i}\hbar \dot{U}(t)U^+(t) = H(t) \qquad U(0) = 1\,. \tag{4.73}$$

In order to proceed we factorize $U(t)$ in the general form (see table 4)

$$U(t) = S_1(\beta_1, a_1)S_2(\beta_2, a_2)S_{12}(\beta, a_1, a_2)T(\tau, a_1, a_2)R_1(\theta_1, a_1)R_2(\theta_2, a_2)\,. \tag{4.74}$$

Introducing this form of $U(t)$ in equation (4.73) we find, after a cumbersome calculation, the following set of 10 strongly coupled first-order nonlinear differential equations. They read as follows:

$$\frac{\mathrm{i}\hbar}{2}\dot{\eta}_1 = g_1 + 2s_1\eta_1 + g_1^*\eta_1^2 + 2(1-|\eta_1|^2)\frac{\eta}{1+|\eta|^2}G_{12} \tag{4.75}$$

$$\frac{\mathrm{i}\hbar}{2}\dot{\eta}_2 = g_2 + 2s_2\eta_2 + g_2^*\eta_2^2 + 2(1-|\eta_2|^2)\frac{\eta}{1+|\eta|^2}G_{12}^* \tag{4.76}$$

$$\frac{\mathrm{i}\hbar}{2}\dot{\eta} = H_{12} + \left[s_1 + s_2 + \mathrm{Re}\{g_1\eta_1^* + g_2\eta_2^*\}\right]\eta + \eta^2 H_{12}^* - 2\eta\frac{\mathrm{Re}\{\eta(G_{12}\eta_1^* + G_{12}^*\eta_2^*)\}}{1+|\eta|^2} \tag{4.77}$$

$$\frac{\mathrm{i}\hbar}{2}\dot{\mu} = \frac{1-|\eta|^2}{1+|\eta|^2}(G_{12} - G_{12}^*\mu^2) + \mu\left[s_1 - s_2 + \mathrm{Re}\{g_1\eta_1^* - g_2\eta_2^*\}\right] - 2\mu\frac{\mathrm{Re}\{\eta(G_{12}\eta_1^* - G_{12}^*\eta_2^*)\}}{1+|\eta|^2} \tag{4.78}$$

$$-\frac{\hbar}{4}\dot{\theta}_1 = s_1 + \mathrm{Re}\{g_1\eta_1^* + H_{12}\eta^*\} - 2\frac{\mathrm{Re}\{G_{12}\eta_1^*\eta\}}{1+|\eta|^2} - \frac{1-|\eta|^2}{1+|\eta|^2}\mathrm{Re}\{G_{12}\mu^*\} \tag{4.79}$$

$$-\frac{\hbar}{4}\dot{\theta}_2 = s_2 + \mathrm{Re}\{g_2\eta_2^* + H_{12}\eta^*\} - 2\frac{\mathrm{Re}\{G_{12}\eta_2\eta^*\}}{1+|\eta|^2} + \frac{1-|\eta|^2}{1+|\eta|^2}\mathrm{Re}\{G_{12}\mu^*\} \tag{4.80}$$

and initial conditions given by

$$\eta_1(0) = \eta_2(0) = \eta(0) = \mu(0) = \theta_1(0) = \theta_2(0) = 0 \tag{4.81}$$

where $G_{12}(\eta_1, \eta_2)$ and $H_{12}(\eta_1, \eta_2)$ are defined as

$$G_{12}(\eta_1, \eta_2) = \frac{e + \eta_1\eta_2^* e^* + \eta_1 d^* + \eta_2^* d}{\sqrt{1-|\eta_1|^2}\sqrt{1-|\eta_2|^2}} \tag{4.82}$$

$$H_{12}(\eta_1, \eta_2) = \frac{d + \eta_1\eta_2 d^* + \eta_1 e^* + \eta_2 e}{\sqrt{1-|\eta_1|^2}\sqrt{1-|\eta_2|^2}}\,. \tag{4.83}$$

One can easily see that these equations contain free Riccati terms appearing together with strongly coupled terms arising mainly from G_{12} and H_{12}. One could, in principle, obtain the functions μ, θ_1 and θ_2, but first the system formed by η_1, η_2 and η must be solved since it constitutes the core of the coupling. A systematic method to derive an approximate solution can of course always be devised. One begins with the obvious assumption $G_{12}^{(0)} = H_{12}^{(0)} = \eta^{(0)} = 0$, and the zeroth-order approximation is obtained ($\eta_1^{(0)}$ and $\eta_2^{(0)}$) by solving the Riccati equations (4.75), (4.76). In the next step one can construct the first-order approximations $G_{12}^{(1)}$, $H_{12}^{(1)}$ and $\eta^{(1)}$ and this iterative process can be continued until the desired degree of approximation is needed. In any case, aside from any approximation, one can deduce from the system the following general conclusions:

- The solution can only be expressed as spin-coherent states if and only if the following condition holds: $g_1 = g_2 = d = 0$.
- The solution can only be expressed as two-mode squeezed states (or $SU(1,1)$ generalized coherent states) if and only if $g_1 = g_2 = e = 0$.
- A solution in terms of tensor products of one-mode squeezed states can be found if $e = d = 0$.
- In spite of the fact that a general analytic solution of the general case does not seem to exist we have found several interesting particular cases. Let us make the ansatz

$$\eta_j = -\mathrm{i} \tanh r_j \exp(\mathrm{i} \arg g_j) \qquad j = 1, 2 \tag{4.84}$$

$$\eta = -\mathrm{i} \tanh r \exp(\mathrm{i} \arg d) \tag{4.85}$$

$$\mu = -\mathrm{i} \tan q \exp(\mathrm{i} \arg e) \,. \tag{4.86}$$

Next we impose the following condition on some of the functions appearing in $H(t)$. In fact, these conditions are

$$\arg g_j = -\frac{4}{\hbar} \int_0^t s_j(t)\, \mathrm{d}t \qquad j = 1, 2 \tag{4.87}$$

$$\arg d = \frac{\arg g_1 + \arg g_2}{2} + \delta_1 \frac{\pi}{2} \tag{4.88}$$

$$\arg e = \frac{\arg g_1 - \arg g_2}{2} + \delta_2 \frac{\pi}{2} \,. \tag{4.89}$$

Under the above particular assumptions we can obtain non-trivial decouplings of the complicated system (4.75)–(4.78) and we find that the initial system reduces to a new one with only real functions and just *four* nonlinear coupled ordinary differential equations:

$$\dot{r}_1 + \dot{r}_2 = \frac{2}{\hbar}(|g_1| + |g_2|) + 2\epsilon_1 \sinh(2r)\dot{q} \qquad r_1(0) = 0 \tag{4.90}$$

$$\dot{r}_1 - \dot{r}_2 = \frac{2}{\hbar}(|g_1| - |g_2|) + 2\epsilon_2 \sinh(2r)\dot{q} \qquad r_2(0) = 0 \tag{4.91}$$

$$\dot{q} \cosh(2r) = \frac{2}{\hbar}\{|e| \cosh(r_1 + \epsilon_3 r_2) + \epsilon_4 |d| \sinh(r_1 + \epsilon_3 r_2)\} \qquad q(0) = 0 \tag{4.92}$$

$$\dot{r} = \frac{2}{\hbar}\{|d| \cosh(r_1 + \epsilon_3 r_2) + \epsilon_4 |e| \sinh(r_1 + \epsilon_3 r_2)\} \qquad r(0) = 0 \,. \tag{4.93}$$

The four simplified cases are summarized in table 6 where the meaning of ϵ_i with $i = 1, 2, 3$ and 4 becomes evident. Cases VI.a, VI.b and VI.d have not been described in the literature, at least to the authors knowledge. Case VI.c can be proved to be equivalent to the one described in [8]. Table 6 represents a complete classification of the types of

Table 6. Reductions of the general case.

Case	δ_1	δ_2	ϵ_1	ϵ_2	ϵ_3	ϵ_4
VI.a	0	−1	0	−1	−1	1
VI.b	−1	0	−1	0	1	1
VI.c	0	1	0	1	−1	−1
VI.d	1	0	1	0	1	−1

systems one can encounter in dealing with the general case, which has in fact been reduced to solve four different coupled real ordinary nonlinear differential equations.

5. Comments and conclusions

In this paper we have considered a general time-dependent quantum harmonic oscillator with two modes which exhibits a dynamical $SO(3,2)$ symmetry. In order to report a complete classification of the exact solutions of this quantum system we have exploited the reduction of the large dynamical group to some of its subgroups $SU(2) \times U(1)$, $SU(1,1) \times U(1)$ and $SU(1,1) \times SU(1,1)$ each of which describe different quantum systems of physical interest. The general formalism in the case of triparametric groups proceeds as follows: one first identifies the time evolution operator among the corresponding subgroup elements, using the exponential mapping, through a functional time-dependent variable that must be a solution (in all cases) of a first-order nonlinear Riccati equation. The instantaneous eigenstates can be put in correspondence with the eigenstates of the two uncoupled one-mode time-dependent quantum harmonic oscillators through a unitary operator. The generator of this transformation is also an element of the subgroup considered and can be non-trivially parametrized through another functional time-dependent variable, but this time just algebraically related to the previous differential variables of the system. We have also found that some cases exhibiting the full $SO(3,2)$ complete dynamical symmetry can be reduced to two one-mode uncoupled time-dependent quantum harmonic oscillators with the help of a suitable unitary transformation. Such particular cases also have an exact solution that can be found with the general methods described here. Finally, the general case has also been considered. Its exact solution is uniquely related to the solution of a system of 10 Riccati-type differential equations, but this time strongly coupled. Imposing a not too restrictive set of conditions on the parametric functions we have been able to solve a great many general Hamiltonians with properties summarized in table 6. Besides the presentation of the general formalism, the study of this case probably constitutes the main result of the paper since the four decoupled cases have been described for the first time with the exception of case VI.c [8]. Much work remains to be done in the field of applications of these solutions to physical Hamiltonians describing laser–matter interaction in various situations. The appropriate place to report on these results will be periodicals dealing with applied physics or quantum optics, but in any case work in this direction is now in progress.

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